

Discrete Riesz MRA on local fields of positive characteristic

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Abstract

We propose a method to construct Riesz MRA on local fields of positive characteristic and corresponding scaling step functions that generate it

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1 Introduction

The simplest example of a local field with positive characteristic is a Vilenkin group. More precisely, Vilenkin group is an additive group of the local field $F^{(s)}$ when $s = 1$. Additive group F^+ of the local field $F^{(s)}$ with positive characteristic is a product \mathfrak{G}^s of Vilenkin groups \mathfrak{G} . Therefore discrete wavelets on local fields is an alternative method for multidimensional discrete data processing.

V. Protasov and Yu. Farkov [7]-[9] obtained the necessary and sufficient conditions under which a refinable function φ generates an orthogonal MRA on Vilenkin group $F^{(1)}$ and indicated some methods for constructing such refinable functions. They proved that the refinable function φ generates an orthogonal MRA if and only if the mask m_0 does not have any blocked sets. The problem of finding the blocked sets is an exhaustive search problem. In articles [19],[20] a new method for constructing refinable step functions is proposed. This method is based on a new concept of N-valid trees. Apparently, this method gives all step functions generating an orthogonal MRA. In [10],[18] some algorithms for constructing biorthogonal compactly supported wavelets on Vilenkin groups are researched and new examples of biorthogonal compactly supported wavelets on Vilenkin groups are given. In [12],[13] 1-valid trees are used for constructing Riesz bases on Vilenkin groups.

The simplest example of a local field with characteristic zero is the field \mathbb{Q}_p of p -adic numbers. Wavelet theory over the field \mathbb{Q}_p is different from the wavelet theory on Vilenkin groups [15]-[17].

In this article we will discuss MRA on local fields of positive characteristic. First results on wavelet analysis on local fields are received by Chinese mathematicians Huikun Jiang, Dengfeng Li, and Ning Jin [1]. They introduced the notion of orthogonal MRA on local fields, for the fields $F^{(s)}$ of positive characteristic p , proved some properties and gave an algorithm for constructing wavelets for a known scaling function. Using these results, they constructed orthogonal MRA and corresponding wavelets for the case when a scaling function is the characteristic function of a unit ball \mathcal{D} . Such MRA is usually called "Haar MRA" and corresponding wavelets are called "Haar wavelets". In the article [4] Biswaranjan Behera and Qaiser Jahanthe proved that a function $\varphi \in L^2(F^{(s)})$ is a scaling function for MRA in $L^2(F^{(s)})$ if and only if

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \text{ for a.e. } \xi \in \mathcal{D}, \quad (1.1)$$

$$\lim_{j \rightarrow \infty} |\hat{\varphi}(\mathfrak{p}^j \xi)| = 1 \text{ for a.e. } \xi \in F^{(s)}, \quad (1.2)$$

and there exists an integral periodic function $m_0 \in L^2(\mathcal{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi) \hat{\varphi}(\mathfrak{p}\xi) \text{ for a.e. } \xi \in F^{(s)} \quad (1.3)$$

where $\{u(k)\}$ is the set of shifts, \mathfrak{p} is a prime element. The condition (1.3) is the necessary condition for inclusion $V_0 \subset V_1$. The condition (1.2) is the necessary and sufficient condition for convergence of the product $\prod m_0(\mathfrak{p}^j \xi)$. The condition (1.1) is the necessary and sufficient condition for orthogonality of the shifts system $\varphi(\xi + u(k))$. It is a difficult problem. If $\hat{\varphi}(\xi) = 1$ on some ball $|\xi| < r$ then we have only two conditions (1.1),(1.3). For such step function methods for constructing orthogonal refinable functions are obtained in the article [14]. B.Behera and Q.Jahan [5] found a condition on the scaling functions φ and $\tilde{\varphi}$ for dual MRAs under which the associated wavelets ψ_l and $\tilde{\psi}_l$ generate the biorthogonal affine systems $(\psi)_{l,j,k}$ and $(\tilde{\psi})_{l,j,k}$ that form Riesz bases for $L_2(F^{(s)})$. Currently, methods for constructing nonorthogonal wavelets on local fields are missing.

In this article we will consider a Riesz MRA with the step scaling functions φ for which $\text{supp } \hat{\varphi}$ is obtained by *spreading* the unit ball K_0^\perp . We will give an algorithm for constructing such Riesz scaling functions. To construct this Riesz scaling functions we will use N-valid trees.

2 Local field of positive characteristic as a vector space over a finite field

In works [1]-[5] authors use the notation and methods of the book by Taibleson [6]. We will use another methods [14].

Let $K = F^{(s)}$ be a local field with positive characteristic p . Its elements are infinite sequences

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

where

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}), a_j^{(\nu)} \in GF(p).$$

Let $\lambda \in GF(p^s)$. Since

$$\begin{aligned} \lambda a &= (\dots \mathbf{0}_{-1}, \lambda, \mathbf{0}_1, \dots) \cdot (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots) = \\ &= (\lambda + \mathbf{0}x + \mathbf{0}x^2 + \dots)(\mathbf{a}_n x^n + \mathbf{a}_{n+1} x^{n+1} + \dots) = \lambda \mathbf{a}_n x^n + \lambda \mathbf{a}_{n+1} x^{n+1} + \dots = \\ &= (\dots \mathbf{0}_{n-1}, \lambda \mathbf{a}_n, \lambda \mathbf{a}_{n+1}, \dots) \end{aligned}$$

it follows that the product λa is defined coordinate-wise. The sum in the field K is defined coordinate-wise also. With such operations $F^{(s)}$ is a vector space. If we define the modulus $|\lambda|$ by the equation

$$|\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

and norm $\|a\|$ by the equation

$$\|a\| = \frac{1}{p^{sn}}, \mathbf{a}_n \neq 0 \quad (2.1)$$

then we can consider the field $F^{(s)}$ as a vector normalized space over the field $GF(p^s)$. Let

$$K_n = \{a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots) : n \in \mathbb{N}, \mathbf{a}_j \in GF(p^s)\}$$

be a ball of radius $\frac{1}{p^{sn}}$. For any $n \in \mathbb{Z}$ choose an element $g_n \in K_n \setminus K_{n+1}$ and fix it. We will call this system $(g_n)_{n \in \mathbb{Z}}$ a basic sequence.

Theorem 2.1 ([14]) *Let $(g_n)_{n \in \mathbb{Z}}$ be a fixed basic sequence in K . Any element $a \in K$ may be written as sum of the series*

$$a = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_n, \bar{\lambda}_n \in GF(p^s). \quad (2.2)$$

It means that the sequence (g_n) is a basis of vector space K . Further we will suppose $g_n = (\dots, \mathbf{0}_{n-1}, \mathbf{1}_n, \mathbf{0}_{n+1}, \dots)$, where $\mathbf{1}_n = (1, 0, \dots, 0)$.

Definition 2.1 *The operator*

$$\mathcal{A} : a = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_n \mapsto \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_{n-1}$$

is called a delation operator.

Remark 1. Since additive group $K^+ = F^{(s)+}$ is Vilenkin group \mathfrak{G} with $\mathfrak{G}_{ns} = F_n^{(s)+}$ it follows that $\mathcal{A}K_n = \mathcal{A}K_{n-1}$ and $\int_{K^+} f(\mathcal{A}u) d\mu = \frac{1}{p^s} \int_{K^+} f(x) d\mu$.

3 Set of characters as vector space over a finite field

Now we define Rademacher function on the vector space K . If

$$a = (\dots, \mathbf{0}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

and

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}), a_j^{(\nu)} \in GF(p)$$

then we define functions $r_n(a) = e^{\frac{2\pi i}{p} a_k^{(l)}}$, where $n = ks + l$, $0 \leq l < s$.

Lemma 3.1 ([14]) *Any character $\chi \in X$ can be expressed uniquely as product*

$$\chi = \prod_{n=-\infty}^{+\infty} r_n^{\alpha_n} \quad (\alpha_n = \overline{0, p-1}), \quad (3.1)$$

in which the number of factors with positive numbers are finite.

If we write the character χ as

$$\chi = \prod_{k \in \mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}$$

and denote

$$\mathbf{r}_k^{\mathbf{a}_k} := r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}$$

where $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$ then we can write the character χ as the product

$$\chi = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}. \quad (3.2)$$

The function $\mathbf{r}_k = \mathbf{r}_k^{(1,0,\dots,0)}$ is called Rademacher function.

Assume by definition

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} := \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}_k}, \quad \mathbf{a}_k, \mathbf{b}_k \in GF(p^s).$$

and

$$\chi^{\mathbf{b}} := \prod_{k \in \mathbb{Z}} (\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}}.$$

Then

$$\mathbf{r}_k^{\mathbf{u} + \mathbf{v}} = \mathbf{r}_k^{\mathbf{u}} \cdot \mathbf{r}_k^{\mathbf{v}}, \quad \mathbf{u}, \mathbf{v} \in GF(p^s)$$

and the set of characters of the field $F^{(s)}$ is a vector space $(X, *, \cdot^{GF(p^s)})$ over the finite field $GF(p^s)$ with product as interior operation and power as exterior operation. It follow from (3.2) that annihilator $(F_k^{(s)})^\perp$ consists of characters $\chi = \mathbf{r}_{k-1}^{\mathbf{a}_{k-1}} \mathbf{r}_{k-2}^{\mathbf{a}_{k-2}} \dots$

The next lemma is the basic property of Rademacher functions on local field with positive characteristic.

Lemma 3.2 ([14]) *Let $g_j = (\dots, \mathbf{0}_{j-1}, (1, 0, \dots, 0)_j, \mathbf{0}_{j+1}, \dots) \in F^{(s)}$, $\mathbf{a}_k, \mathbf{u} \in GF(p^s)$. Then $(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u} g_j) = 1$ for any $k \neq j$.*

4 Riesz MRA on local fields of positive characteristic

Denote

$$H_0 = \{a : a = \mathbf{a}_{-1} g_{-1} + \dots + \mathbf{a}_{-\nu} g_{-\nu}, \quad \nu \in \mathbb{N}, \quad \mathbf{a}_{-j} \in GF(p^s)\},$$

$$H_0^{(\nu)} = \{a : a = \mathbf{a}_{-1} g_{-1} + \dots + \mathbf{a}_{-\nu} g_{-\nu}, \quad \mathbf{a}_{-j} \in GF(p^s)\}, \quad \nu \in \mathbb{N}.$$

H_0 is an analog of the set $(N_0)^s = N_0 \times \dots \times N_0$.

Define a dilation operator \mathcal{A} on the set of characters by the equation $(\chi \mathcal{A}, x) = (\chi, \mathcal{A}x)$. It is evident that $\mathbf{r}_j \mathcal{A} = \mathbf{r}_{j+1}$, $(K_n^+)^{\perp} \mathcal{A} = (K_{n+1}^+)^{\perp}$ and $\int_X f(\chi \mathcal{A}) d\nu = \frac{1}{p^s} \int_X f(\chi) d\nu$ [14].

We will use next properties of annihilators [14].

$$1) \quad \int_{(K_n^+)^{\perp}} (\chi, x) d\nu(\chi) = p^{sn} \mathbf{1}_{K_n^+}(x),$$

$$2) \quad \int_{K_n^+} (\chi, x) d\mu(x) = \frac{1}{p^{sn}} \mathbf{1}_{(K_n^+)^{\perp}}(\chi).$$

3) Let $\chi_{n,l} = \mathbf{r}_n^{\mathbf{a}_n} \mathbf{r}_{n+1}^{\mathbf{a}_{n+1}} \dots \mathbf{r}_{n+l}^{\mathbf{a}_{n+l}}$ be a character which does not belong to $(K_n^+)^{\perp}$. Then

$$\int_{(K_n^+)^{\perp} \chi_{n,l}} (\chi, x) d\nu(\chi) = p^{ns} (\chi_{n,l}, x) \mathbf{1}_{K_n^+}(x).$$

4) Let $h_{n,l} = \mathbf{a}_{n-1}g_{n-1} + \mathbf{a}_{n-2}g_{n-2} + \dots + \mathbf{a}_{n-l}g_{n-l} \notin K_n^+$. Then

$$\int_{K_n^+ + h_{n,l}} (\chi, x) d\mu(x) = \frac{1}{p^{ns}} (\chi, h_{n,l}) \mathbf{1}_{(K_n^+)^{\perp}}(\chi).$$

Definition 4.1 *A family of closed subspaces V_n , $n \in \mathbb{Z}$, is said to be a Riesz multiresolution analysis of $L_2(K)$ if the following axioms (conditions) are satisfied:*

A1) $V_n \subset V_{n+1}$;

A2) $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L_2(K)$ and $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;

A3) $f(x) \in V_n \iff f(\mathcal{A}x) \in V_{n+1}$ (\mathcal{A} is a dilation operator);

A4) $f(x) \in V_0 \implies f(x \dot{-} h) \in V_0$ for all $h \in H_0$;

A5) *there exists a function $\varphi \in L_2(K)$ such that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ is a Riesz basis for V_0 .*

A function φ occurring in axiom A5 is called a scaling function.

We recall that the family $(f_j) \subset L_2(K)$ is called a Riesz system with constants A and B ($A, B > 0$) if, for every sequence $C = (c_j) \in l_2$ the series $\sum_j c_j f_j$ converges in $L_2(K)$ and

$$A\|C\|_{l_2}^2 \leq \left\| \sum c_n \varphi_n \right\|_{L_2(G)}^2 \leq B\|C\|_{l_2}^2. \quad (4.1)$$

Lemma 4.1 ([4], Theorem 4.1.) *Let $(V_n)_{n \in \mathbb{Z}}$ be a sequence of closed subspaces of $L_2(K)$ satisfying conditions (A1), (A3) and (A5) of Definition 4.1. Then, $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$.*

Lemma 4.2 ([4], Theorem 4.2.) *Let $\hat{\varphi} \in L_2(X)$ be a continuous function at the point $\chi = 1$ and $\hat{\varphi}(1) \neq 0$. Suppose also that $(V_n)_{n \in \mathbb{Z}}$ is a sequence of closed subspaces of $L_2(K)$ satisfying conditions (A1), (A3) and (A5) of Definition 4.1. Then $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L_2(K)$.*

Next we will follow the conventional approach. Let $\varphi(x) \in L_2(K)$, and suppose that $(\varphi(x \dot{-} h))_{h \in H_0}$ form a Riesz basis in the closure of their linear hull in the norm $L_2(K)$. With the function φ and the dilation operator \mathcal{A} , we define subspaces $V_n = \overline{\text{span}(\varphi(\mathcal{A}^n x \dot{-} h))_{h \in H_0}}$ closed in $L_2(K)$. If the

family $(V_n)_{n \in \mathbb{Z}}$ is an MRA, we will say that the function φ generates MRA. It is clear that the system $(p^{\frac{ns}{2}} \varphi(\mathcal{A}^n x - h))_{h \in H_0}$ is a Riesz basis for V_n and $f(x) \in V_n$ if and only if $f(\mathcal{A}x) \in V_{n+1}$. We want to propose an algorithm for constructing a function φ that generate a Riesz MRA and corresponding wavelets.

We shall assume that the function φ generating a Riesz MRA satisfies the inequality

$$0 < A \leq |\hat{\varphi}(\chi)|^2 \leq B, \quad (4.2)$$

on some set $E \subset X$ of measure $\nu E = 1$ that is obtained by *spreading* the set K_0^\perp . We now give a precise characterization of E .

Definition 4.2 Let $K = F^{(s)}$ be a local field of characteristic p , X -group of additive characters for K^+ , $N \in \mathbb{N}$, $M \in \mathbb{N}_0 = \mathbb{N} \sqcup \{0\}$. A set $E \subset X$ is said to be (N, M) -elementary if it is the disjoint union of p^{Ns} cosets of the form

$$(K_{-N}^+)^\perp \zeta_j = (K_{-N}^+)^\perp \underbrace{\mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\bar{\alpha}_{-1}}}_{\xi_j} \underbrace{\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}}_{\eta_j} = (K_{-N}^+)^\perp \xi_j \eta_j,$$

for $j = 0, 1, \dots, p^{Ns} - 1$ such that the following conditions hold

- 1) $\bigsqcup_{j=0}^{p^{Ns}-1} (K_{-N}^+)^\perp \xi_j = (K_0^+)^\perp$, $(K_{-N}^+)^\perp \xi_0 = (K_{-N}^+)^\perp$.
- 2) For every $l = \overline{0, M+N-1}$ we have $((K_{-N+l+1}^+)^\perp \setminus (K_{-N+l}^+)^\perp) \cap E \neq \emptyset$.

So, to obtain an (N, M) -elementary set, we shift any coset

$(K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\bar{\alpha}_{-1}}$ per the unique element $\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}$ that any difference $((K_{-N+l+1}^+)^\perp \setminus (K_{-N+l}^+)^\perp)$ contains at least one shift.

Lemma 4.3 ([14]) The set $H_0 \subset K$ is a total orthonormal system on any (N, M) -elementary set $E \subset X$.

Lemma 4.4 Let $K = F^{(s)}$ be a local field of characteristic p , $E \subset (K_M^+)^\perp$ an (N, M) -elementary set in X , $\varphi \in L^2(K)$, $\text{supp } \hat{\varphi} = E$, $A, B > 0$. The system of shifts $(\varphi(x - h))_{h \in H_0}$ is a Riesz system with constants A and B if and only if

$$A \leq |\hat{\varphi}(\chi)|^2 \leq B,$$

a.e. on E .

Proof. S u f f i c i e n c y. First, we find an upper bound for $\left\| \sum_{h \in \tilde{H}_0} c_h \varphi(x \dot{-} h) \right\|_2^2$ assuming that $\tilde{H}_0 \subset H_0$ is a finite set. Using Plancherel's equality, we have

$$\begin{aligned} \int_K \left| \sum_{h \in \tilde{H}_0} c_h \varphi(x \dot{-} h) \right|^2 d\mu(x) &= \int_X \left| \sum_{h \in \tilde{H}_0} c_h \hat{\varphi}(\chi) \overline{(\chi, h)} \right|^2 d\nu(\chi) \leq \\ B \int_E \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi, h)} \right|^2 d\nu(\chi) &= B \sum_{j=0}^{p^{Ns}-1} \int_{(K_{-N}^+)^{\perp} \zeta_j} \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi, h)} \right|^2 d\nu(\chi). \end{aligned}$$

We rewrite the inner integral using invariance of the integral

$$\begin{aligned} \int_{(K_{-N}^+)^{\perp} \zeta_j} \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi, h)} \right|^2 d\nu(\chi) &= \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi) \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi, h)} \right|^2 d\nu(\chi) = \\ &= \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi \eta_j) \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi \eta_j, h)} \right|^2 d\nu(\chi) = \\ &= \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \xi_j}(\chi) \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi \eta_j, h)} \right|^2 d\nu(\chi) = \\ &= \int_{(K_{-N}^+)^{\perp} \xi_j} \sum_{h \in \tilde{H}_0} \sum_{g \in \tilde{H}_0} c_h \overline{c_g} \overline{(\eta_j, h)} (\eta_j, g) \overline{(\chi, h)} (\chi, g) d\nu(\chi). \end{aligned}$$

Since

$$\begin{aligned} (\eta_j, h) &= (\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}, a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \dots \dot{+} a_{-\nu}g_{-\nu}) = 1, \\ (\eta_j, g) &= (\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}, b_{-1}g_{-1} \dot{+} b_{-2}g_{-2} \dot{+} \dots \dot{+} b_{-\nu}g_{-\nu}) = 1, \end{aligned}$$

we get

$$\sum_{j=0}^{p^N-1} \int_{(K_{-N}^+)^{\perp} \zeta_j} \left| \sum_{h \in \tilde{H}_0} c_h \overline{(\chi, h)} \right|^2 d\nu(\chi) = \sum_{g, h \in \tilde{H}_0} c_h \overline{c_g} \sum_{j=0}^{p^N-1} \int_{(K_{-N}^+)^{\perp} \xi_j} \overline{(\chi, h)} (\chi, g) d\nu(\chi) =$$

$$= \sum_{g, h \in \tilde{H}_0} c_h \overline{c_g} \int_{(K_0^+)^{\perp}} \overline{(\chi, h)}(\chi, g) d\nu(\chi) = \sum_{h \in \tilde{H}_0} |c_h|^2,$$

and, therefore,

$$\left\| \sum_{h \in \tilde{H}_0} c_h \varphi(x \dot{+} h) \right\|_2^2 \leq B \sum_{h \in \tilde{H}_0} |c_h|^2.$$

Similarly we obtain that

$$\left\| \sum_{h \in \tilde{H}_0} c_h \varphi(x \dot{-} h) \right\|_2^2 \geq A \sum_{h \in \tilde{H}_0} |c_h|^2.$$

N e c e s s i t y. Let $(\varphi(x \dot{-} h))_{h \in H_0}$ be a Riesz system with bounds A and B i.e. for any $(c_h) \in l_2$ the inequality

$$A \sum_{h \in H_0} |c_h|^2 \leq \left\| \sum_{h \in H_0} c_h \varphi(x \dot{-} h) \right\|_{L_2(K)}^2 \leq B \sum_{h \in H_0} |c_h|^2 \quad (4.3)$$

holds. It follows that

$$f(x) = \sum_{h \in H_0} c_h \varphi(x \dot{-} h) \in L_2(K).$$

By Plancherel's equality we have

$$\left\| \sum_{h \in H_0} c_h \varphi(x \dot{-} h) \right\|_{L_2(K)}^2 = \left\| \sum_{h \in H_0} c_h \hat{\varphi}(\chi) \overline{(\chi, h)} \right\|_{L_2(E)}^2.$$

Therefore we can write the equality (4.3) in the form

$$A \sum_{h \in H_0} |c_h|^2 \leq \hat{\varphi}(\chi) \left\| \sum_{h \in H_0} c_h \overline{(\chi, h)} \right\|_{L_2(E)}^2 \leq B \sum_{h \in H_0} |c_h|^2.$$

Denote $g(\chi) = \sum_{h \in H_0} c_h \overline{(\chi, h)}$. Since (H_0) is an orthonormal system in $L_2(E)$ we obtain

$$A \|g\|_{L_2(E)}^2 \leq \int_E |\hat{\varphi}(\chi)|^2 |g(\chi)|^2 d\nu(\chi) \leq B \|g\|_{L_2(E)}^2$$

or another

$$A \leq \int_E |\hat{\varphi}(\chi)|^2 \frac{|g(\chi)|^2}{\|g\|^2} d\nu(\chi) \leq B.$$

It follows that for any $h \in L_2(E)$ with norm $\|h\|_{L_2(E)} = 1$

$$A \leq \int_E |\hat{\varphi}(\chi)|^2 |h(\chi)| d\nu(\chi) \leq B. \quad (4.4)$$

Therefore $A \leq \text{vrai sup} |\hat{\varphi}(\chi)|^2 \leq B$.

Let us show that $A \leq |\hat{\varphi}(\chi)|^2$ a.e. in E . Assume the converse. Then there exists $E_1 \subset E$ such that $\nu(E_1) > 0$ and $|\hat{\varphi}(\chi)|^2 < A - \varepsilon$ on the set E_1 . Taking $h(x) = \frac{\mathbf{1}_{E_1}(\chi)}{\nu(E_1)}$ we have

$$\int_E |\hat{\varphi}(\chi)|^2 |h(\chi)| d\nu(\chi) = \int_{E_1} |\hat{\varphi}(\chi)|^2 \frac{1}{\nu(E_1)} d\nu(\chi) \leq A - \varepsilon < A.$$

But this contradicts inequality 4.4. \square

Lemma 4.5 *Let $K = F^{(s)}$ be a local field, $E \subset X$ an (N, M) elementary set in X , $\text{supp } \hat{\varphi} = \text{supp } \hat{\psi} = E$.*

1) *If $\hat{\varphi}(\chi)\hat{\psi}(\chi) = 1$ a.e. on E , then $(\varphi(x \dot{-} h), \psi(x \dot{-} g))_{h,g \in H_0}$ is a biorthonormal system on E .*

2) *Conversely, if $(\varphi(x \dot{-} h), \psi(x \dot{-} g))_{h,g \in H_0}$ is a biorthonormal system on E , then $\hat{\varphi}(\chi)\hat{\psi}(\chi) = 1$ a.e. on E .*

Proof. 1) Using Plancherel's equality and Lemma 4.1, we have

$$\int_K \varphi(x \dot{-} h) \overline{\psi(x \dot{-} g)} d\mu(x) = \int_E \overline{(\chi, h)}(\chi, g) d\nu(\chi) = \delta_{h,g}.$$

2) Using Plancherel's equality and a biorthogonality of the system $(\varphi(x \dot{-} h), \psi(x \dot{-} h))$ we have

$$\int_K \overline{\varphi(x \dot{-} h)} \psi(x) d\mu(x) = \int_E \overline{\hat{\varphi}(\chi)} \hat{\psi}(\chi)(\chi, h) d\nu(\chi) = c_h = \begin{cases} 1, & h = 0 \\ 0 & h \neq 0, h \in H_0 \end{cases}$$

By the uniqueness theorem $\overline{\hat{\varphi}(\chi)} \hat{\psi}(\chi) = 1$ a.e. on E . \square

The following lemma obviously follows from the equality $\int_K f(\mathcal{A}x) d\mu =$

$$\frac{1}{p^s} \int_K f(x) d\mu.$$

Lemma 4.6 *Let $n \in \mathbb{N}$. The shift system $(\varphi(x \dot{-} h))_{h \in H_0}$ is a Riesz system with constants A and B if and only if the system $(p^{\frac{ns}{2}} \varphi(\mathcal{A}^n x \dot{-} h))_{h \in H_0}$ is a Riesz system with constants A and B .*

Recall that $V_n = \overline{\text{span}(\varphi(\mathcal{A}^n x \dot{-} h))_{h \in H_0}}$.

Lemma 4.7 *Suppose that $\varphi \in L_2(K)$, E is an (N, M) -elementary set, $\text{supp } \hat{\varphi} = E$, $\hat{\varphi}$ satisfies the conditions (4.2) on E . Then $V_0 \subset V_1$ if and only if the function φ satisfies the equation*

$$\varphi(x) = \sum_{h \in H_0} \beta_h \varphi(\mathcal{A}x \dot{-} h), \quad \sum_{h \in H_0} |\beta_h|^2 < +\infty. \quad (4.5)$$

Proof. N e c e s s i t y. By lemma 4.4 $\varphi(x \dot{-} h)$ is a Riesz system with constants A and B . Taking Lemma 4.6 into account, we get that $\sqrt{p^s}(\varphi(\mathcal{A}x \dot{-} h))_{h \in H_0}$ is a Riesz system with constants A and B , and, therefore, form a basis of V_1 . Since $\varphi \in V_0 \subset V_1$, the equation (4.5) holds and we have $\sum |\beta_h|^2 < +\infty$.

S u f f i c i e n c y. Let $f \in \text{span}(\varphi(x \dot{-} h))_{h \in H_0}$, i.e.

$$f(x) = \sum_{\tilde{h} \in \tilde{H} \subset H_0} \alpha_{\tilde{h}} \varphi(x \dot{-} \tilde{h}), \quad (4.6)$$

where $\tilde{H} \subset H_0$ is the finite set. Substituting (4.5) in (4.6), we obtain

$$f(x) = \sum_{\tilde{h} \in \tilde{H}} \alpha_{\tilde{h}} \sum_{h \in H_0} \beta_h \varphi(\mathcal{A}x \dot{-} (\mathcal{A}\tilde{h} \dot{+} h)) = \sum_{h \in H_0} \sum_{\tilde{h} \in \tilde{H}} \alpha_{\tilde{h}} \beta_h \varphi(\mathcal{A}x \dot{-} (\mathcal{A}\tilde{h} \dot{+} h)). \quad (4.7)$$

Since the set H_0 is a group and $\mathcal{A}\tilde{h} \in H_0$, it follows that $f(x) \in V_1$ \square

Therefore we need to look for a function $\varphi \in L_2(K)$, that generates an MRA in $L_2(K)$, as a solution of the refinement equation (4.5). A solution of the refinement equation (4.5) is called a *refinable function*.

Theorem 4.1 *Suppose that $\varphi \in L_2(K)$, $E \subset X$ is an (N, M) -elementary set, $\text{supp } \hat{\varphi} = E$, $\hat{\varphi}$ satisfies the conditions (4.2) on E . Then $V_n \subset V_{n+1}$ if and only if the function φ satisfies the equation (4.5)*

This theorem follows from lemmas 4.6-4.7.

Theorem 4.2 *Let $\hat{\varphi} \in L_2(X)$ be a continuous function at the point $\chi = 1$ and $\hat{\varphi}(1) \neq 0$. Suppose that $E \subset X$ is an (N, M) -elementary set, $\text{supp } \hat{\varphi} = E$, $\hat{\varphi}$ satisfies the conditions (4.2) and (4.5). Then φ generates an Riesz MRA.*

Proof. The property A5) follows from lemma 4.4. The property A4) is true, as the set H_0 is a group. The property A3) is evident. The property A1) follows from theorem 4.1. The property A2) follows from lemmas 4.1. and 4.2. \square

5 Construction of scaling function

In this section we will construct functions $\hat{\varphi}$ for which the conditions of theorem 4.2 are satisfied. The refinement equation (4.5) may be written in the form

$$\hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}), \quad (5.1)$$

where

$$m_0(\chi) = \frac{1}{p^s} \sum_{h \in H_0} \beta_h \overline{(\chi \mathcal{A}^{-1}, h)} \quad (5.2)$$

is a mask of the equation (4.4). We will use equation (5.1) to construct the refinable function φ . First we will construct the support of $\hat{\varphi}$ using a concept of N -valid tree [19].

Definition 5.1 *Let T be a tree directed from the root on the set of nodes $GF(p^s)$. The tree T is called as N -valid if the following properties are valid:*

- a) The nodes of this tree are elements $\bar{\alpha} \in GF(p^s)$*
- b) The root of T is $\mathbf{0} = (0, 0, \dots, 0)$*
- c) For any $j = 0, 1, \dots, N - 1$ the set T_j of level j nodes is the set $\{\mathbf{0}\}$*
- d) Any path $(\bar{\alpha}_k \rightarrow \bar{\alpha}_{k+1} \rightarrow \dots \rightarrow \bar{\alpha}_{k+N-1})$ of length $N - 1$ is present in the tree T exactly one time.*

For example for $p = s = N = 2$ we can construct the tree

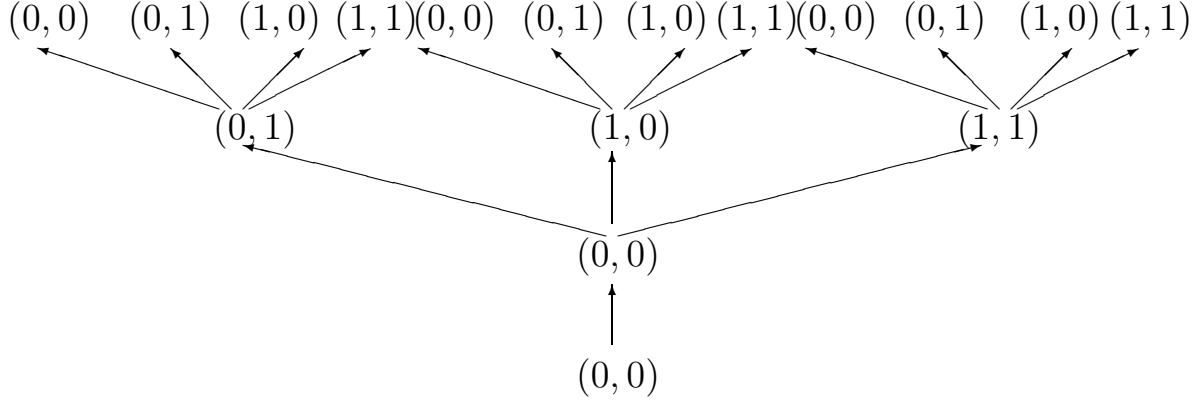


Figure 1

Here we give a method for construction of N -valid trees for any N, s, p . First we construct a *basic tree* T_B of smallest height. Let $\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{p^s-1}$ be all elements of the finite field $GF(p^s)$, and $\bar{\alpha}_0 = (0, 0, \dots, 0)$. We construct a basic tree T_B in the following way.

1. Choose a path $(\bar{\alpha}_0 \rightarrow \bar{\alpha}_0 \rightarrow \dots \rightarrow \bar{\alpha}_0)$ of length $N - 1$. This path contains N nodes $\bar{\alpha}_0$ and the level of the last node is $N - 1$.
2. Connect all elements $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{p^s-1}$ to the last node. We get a tree of height $H = N$ in which any path of length $N - 1$ is present not more than once. But not all paths of length $N - 1$ are present in this tree. This tree is shown in Fig. 2.

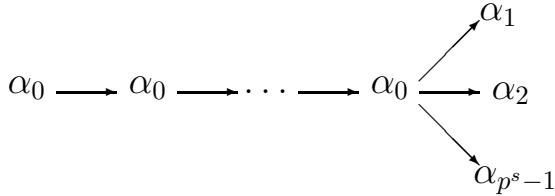


Figure 2. Tree after 2 steps

3. Now we connect all elements $\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{p^s-1}$ to every node $(\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{p^s-1})$ of level N and get a tree of height $H = N + 1$. We can see this tree on Fig.3. After $(N + 1)$ -th step we obtain the N -valid tree T_B of smallest height.

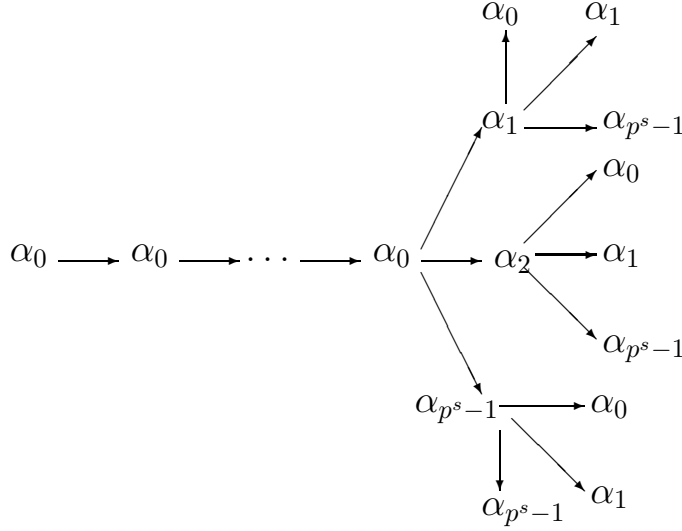


Figure 3. Tree after 3 steps

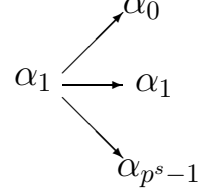


Figure 4. Subtree

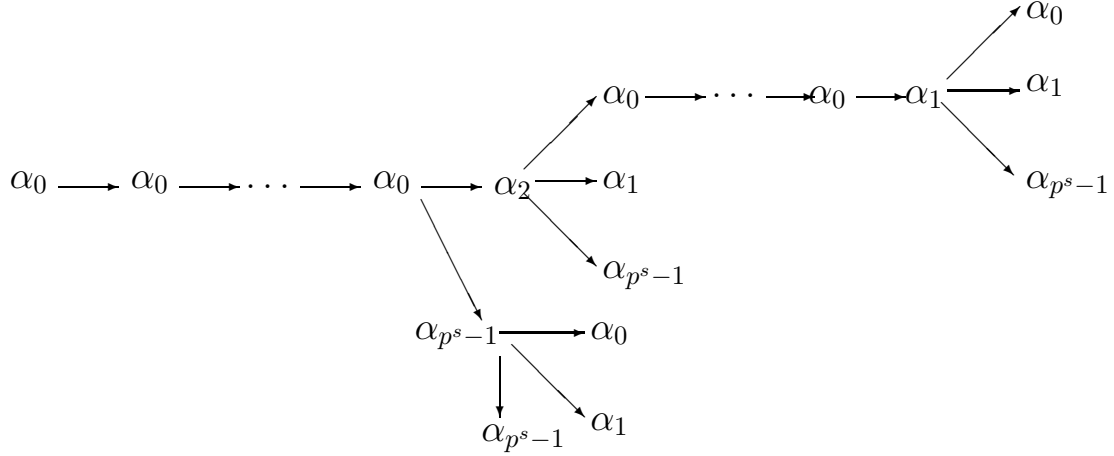


Figure 5. Tree after moving

To obtain another N -valid trees we introduce the concept of *basic step* in the following way.

1. Let T be a N -valid tree. Take a subtree $T_{j(N-1+\nu), N-1+\nu}$ with a node $\alpha_{j(N-1+\nu)}^{(N-1+\nu)}$, $\nu \geq 1$ of the level $N-1+\nu$ as a root. (see figures 3 and 4)
2. Take a path

$$\alpha_{j(\nu)}^{(\nu)} = \alpha_{j(N-1+\nu-N+1)}^{(N-1+\nu-N+1)} \rightarrow \cdots \rightarrow \alpha_{j(N-1+\nu-1)}^{(N-1+\nu-1)} \rightarrow \alpha_{j(N-1+\nu)}^{(N-1+\nu)}$$

of length $N-1$ which ends in this node .

3. Find a path of the length $N-2$ which ends in leaf $\alpha = \alpha_{j(N-1+\nu-1)}^{(N-1+\nu-1)}$ and which coincides with the path

$$\alpha_{j(N-1+\nu-N+1)}^{(N-1+\nu-N+1)} \rightarrow \cdots \rightarrow \alpha_{j(N-1+\nu-1)}^{(N-1+\nu-1)}$$

4. Move the subtree $T_{j(N-1+\nu), N-1+\nu}$ to the leaf $\alpha = \alpha_{j(N-1+\nu-1)}^{(N-1+\nu-1)}$. See Fig.5

If the original tree was N -valid, then after the employment of the basic step we obtain a N -valid tree again. Thus, applying the basic algorithm to the basic tree finite number of times, we will obtain different N -valid trees.

Let T be a N -valid tree. Choose any path of length greater than $N - 1$

$$P = \mathbf{0}_1 \rightarrow \cdots \rightarrow \mathbf{0}_N \rightarrow \bar{\alpha}_\nu \rightarrow \bar{\alpha}_{\nu-1} \rightarrow \cdots \rightarrow \bar{\alpha}_{-N+1} \rightarrow \bar{\alpha}_{-N} =$$

$$= \bar{\alpha}_{\nu+N} \rightarrow \bar{\alpha}_{\nu+N-1} \rightarrow \cdots \rightarrow \bar{\alpha}_{\nu+1} \rightarrow \bar{\alpha}_\nu \rightarrow \bar{\alpha}_{\nu-1} \rightarrow \cdots \rightarrow \bar{\alpha}_{-N+1} \rightarrow \bar{\alpha}_{-N}.$$

Since the tree T is N -valid it follows that $\bar{\alpha}_\nu \neq 0$. Let us construct cosets

$$\begin{aligned} & (K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \cdots \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0}, \\ & (K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_{-N+1}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+2}} \cdots \mathbf{r}_{-1}^{\bar{\alpha}_0} \mathbf{r}_0^{\bar{\alpha}_1}, \\ & \dots\dots\dots \\ & (K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_\nu} \mathbf{r}_{-N+1}^{\bar{\alpha}_{\nu+1}} \cdots \mathbf{r}_{-1}^{\bar{\alpha}_{\nu+N-1}} \mathbf{r}_0^{\bar{\alpha}_{\nu+N}} \end{aligned} \tag{5.3}$$

and denote the union of all such cosets as \tilde{E} .

Lemma 5.1 \tilde{E} is $(N, 1)$ -elementary set.

Proof. Since T is N -valid tree it follows that for any coset

$$(K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_\nu} \mathbf{r}_{-N+1}^{\bar{\alpha}_{\nu+1}} \cdots \mathbf{r}_{-1}^{\bar{\alpha}_{\nu+N-1}} \subset (K_0^+)^\perp$$

there exists unique shift

$$(K_{-N}^+)^\perp \mathbf{r}_{-N}^{\bar{\alpha}_\nu} \mathbf{r}_{-N+1}^{\bar{\alpha}_{\nu+1}} \cdots \mathbf{r}_{-1}^{\bar{\alpha}_{\nu+N-1}} \mathbf{r}_0^{\bar{\alpha}_{\nu+N}} \subset (K_1^+)^\perp.$$

It means that \tilde{E} is $(N, 1)$ -elementary set. \square

Definition 5.2 We say that the set \tilde{E}_X is a periodic extension of \tilde{E} if

$$\tilde{E}_X = \bigcup_{s=1}^{\infty} \bigsqcup_{\bar{\alpha}_1, \dots, \bar{\alpha}_s=0}^{p-1} \tilde{E} \mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \cdots \mathbf{r}_\nu^{\bar{\alpha}_\nu}.$$

If $\bigcap_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n = E$ then we say that \tilde{E} generates this set E , and the N -valid tree generates E also.

Lemma 5.2 *Let T be a N -valid tree of height H . Then*

$$\prod_{n=0}^{\infty} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = \prod_{n=0}^{H-N+1} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n})$$

if $\chi \in (K_{H-2N+2}^+)^{\perp}$.

Proof. Since $\tilde{E}_X \supset (K_{-N}^+)^{\perp}$ and $(K_l^+)^{\perp} \mathcal{A} = (K_{l+1}^+)^{\perp}$ it follows that

$$\mathbf{1}_{\tilde{E}_X}((K_{H-2N+2}^+)^{\perp} \mathcal{A}^{-H+N-2}) = \mathbf{1}_{\tilde{E}_X}((K_{-N}^+)^{\perp}) = 1$$

and the lemma is proved. \square

Lemma 5.3 *Let T be a N -valid tree of height H . Suppose the tree T generates the set $E \subset X$. Then E is an $(N, H - 2N + 1)$ -elementary set.*

Proof. Let us denote

$$m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi), \quad M(\chi) = \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}).$$

First we note that $M(\chi) = \mathbf{1}_E(\chi)$. Indeed

$$\begin{aligned} \mathbf{1}_E(\chi) = 1 &\Leftrightarrow \chi \in E \Leftrightarrow \forall n, \chi \mathcal{A}^{-n} \in \tilde{E}_X \Leftrightarrow \forall n, \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow \\ &\forall n, m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow M(\chi) = 1. \end{aligned}$$

Now we will prove, that $\mathbf{1}_E(\chi) = 0$ for $\chi \in (K_{H-2N+2}^+)^{\perp} \setminus (K_{H-2N+1}^+)^{\perp}$. In another words we need to prove that

$$\mathbf{1}_E(K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{H-2N+1}^{\bar{\alpha}_{H-2N+1}} = 0$$

for $\bar{\alpha}_{H-2N+1} \neq 0$.

By the definition of cosets (5.3), $m((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_0^{\bar{\alpha}_0}) \neq 0$ if and only if the vector $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{-N+1}, \bar{\alpha}_{-N})$ is a path $(\bar{\alpha}_0 \rightarrow \bar{\alpha}_1 \rightarrow \dots \rightarrow \bar{\alpha}_{-N+1} \rightarrow \bar{\alpha}_{-N})$ of the tree T .

Since \tilde{E}_X is a periodic extension of \tilde{E} it follows that the function $m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi)$ is periodic with any period $\mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_{\nu}^{\bar{\alpha}_{\nu}}$, $\nu \in \mathbb{N}$, i.e. $m(\chi \mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_{\nu}^{\bar{\alpha}_{\nu}}) = m(\chi)$ when $\chi \in (K_1^+)^{\perp}$. Using this fact we can write $M(\chi)$ for $\chi \in (K_{H-2N+2}^+)^{\perp} \setminus (K_{H-2N+1}^+)^{\perp}$ in the form

$$M((K_{-N}^+)^{\perp} \zeta) = M((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{H-2N+1}^{\bar{\alpha}_{H-2N+1}}) =$$

[illegible]

Assume that $M((K_{-N}^+)^{\perp}\zeta) \neq 0$. Then all factors in (5.4) are nonzero. So we have the path

$$\mathbf{0} \rightarrow \cdots \rightarrow \mathbf{0} \rightarrow \bar{\alpha}_{H-2N+1} \neq \mathbf{0} \rightarrow \bar{\alpha}_{H-2N} \rightarrow \cdots \rightarrow \bar{\alpha}_0 \rightarrow \cdots \rightarrow \bar{\alpha}_{-N+1} \rightarrow \bar{\alpha}_{-N},$$

where there are N zeroes at the beginning of the path. The length of such path is $H + 1$, which contradicts the condition height of T equals H .

Now we prove that E is $(N, H - 2N + 1)$ elementary set. Since the tree T is N -valid, it has all possible combinations of N elements $\bar{\alpha}_i \in GF(p^s)$ as its paths, and we have the first property of elementary sets satisfied. Also, since height of T is H , there exists a path

$$\bar{\alpha}_1 = \mathbf{0} \rightarrow \cdots \rightarrow \bar{\alpha}_N = \mathbf{0} \rightarrow \bar{\alpha}_{N+1} \neq \mathbf{0} \rightarrow \bar{\alpha}_{N+2} \rightarrow \cdots \rightarrow \bar{\alpha}_{H+1}$$

of length H . Such path generates cosets

$$(K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{N+l}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{N+l-1}} \dots \mathbf{r}_{-N+l-1}^{\bar{\alpha}_{N+1}} \subset (K_{-N+l}^+)^{\perp} \setminus (K_{-N+l-1}^+)^{\perp}$$

for all $l = 1, 2, \dots, H + 1 - N$. Thus we can conclude that E is $(N, H - 2N + 1)$ -elementary set and the lemma is proved. \square .

Now we can formulate an algorithm for constructing the refinable function that generates Riesz MRA.

RF-algorithm.

1. Construct N -valid tree T of height $H \geq 1$ using basic N -valid tree and basic steps.
2. Construct the set $\tilde{E} \subset (K_1^+)^{\perp}$ using formulas (5.3).
3. Construct the function $m_0(\chi)$ on the set $(K_1^+)^{\perp}$ such that
 - 3.1. $m_0((K_N^+)^{\perp}) = 1$,
 - 3.2. $\text{supp } m_0(\chi) = \tilde{E}$,
 - 3.3. $0 < A \leq |m_0(\chi)|^2 \leq B$,
4. Extend the function m_0 periodically with any period $\mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_{\nu}^{\bar{\alpha}_{\nu}}$. It is evident that $\text{supp } m_0(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi)$ and $A \leq |m_0(\chi)|^2 \leq B$.
5. Set $\hat{\varphi}(\chi) = \prod_{n=0}^{H-N+1} m_0(\chi \mathcal{A}^{-n})$

Remark. The Fourier transform $\hat{\varphi}$ may be calculated in the following way. Take any path

$$\bar{\alpha}_{\nu+N} \rightarrow \bar{\alpha}_{\nu+N-1} \rightarrow \cdots \rightarrow \bar{\alpha}_{\nu+1} \rightarrow \bar{\alpha}_{\nu} \rightarrow \bar{\alpha}_{\nu-1} \rightarrow \cdots \rightarrow \bar{\alpha}_{-N}$$

of the length $\geq N$ in that $\bar{\alpha}_{\nu+N} = \bar{\alpha}_{\nu+N-1} = \cdots = \bar{\alpha}_{\nu+1} = 0, \bar{\alpha}_{\nu} \neq 0$. Then we set

$$\begin{aligned} & \hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \cdots \mathbf{r}_{\nu}^{\bar{\alpha}_{\nu}} \cdots \mathbf{r}_{\nu+N}^{\bar{\alpha}_{\nu+N}}) = \\ & = m_0(\mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \cdots \mathbf{r}_0^{\bar{\alpha}_0}) m_0(\mathbf{r}_{-N}^{\bar{\alpha}_{-N+1}} \cdots \mathbf{r}_0^{\bar{\alpha}_1}) \cdots m_0(\mathbf{r}_{-N}^{\bar{\alpha}_{\nu}} \cdots \mathbf{r}_0^{\bar{\alpha}_{\nu+N}}) \end{aligned}$$

Theorem 5.1 *The function φ generates Riesz MRA with constants A^{H-N+2} and B^{H-N+2} .*

Proof. It is evident that $A^{H-N+2} \leq |\hat{\varphi}(\chi)|^2 \leq B^{H-N+2}$ and $\hat{\varphi}((K_{-N}^+)^{\perp}) = 1$. By lemma 5.3

$$\text{supp} \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n}) = E = \bigcap_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n \subset (K_{H-2N+1}^+)^{\perp},$$

so that

$$\text{supp} \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n}) = 0$$

on the set $(K_{H-2N+2}^+)^{\perp} \setminus (K_{H-2N+1}^+)^{\perp}$. Consequently, by lemma 5.2

$$\text{supp} \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n}) = \text{supp} \prod_{n=0}^{H-N+1} m_0(\chi \mathcal{A}^{-n}) = \hat{\varphi}(\chi).$$

So, by theorem 4.2 the function φ generates Riesz MRA. \square

6 Construction of Riesz wavelets

In this section we will give an algorithm for constructing wavelets. We will use the result of B. Behera and Q. Jahan [5], which we formulate in our notations.

Let $\{V_j\}$ and $\{\tilde{V}_j\}$ be biorthogonal MRAs with scaling functions $\varphi, \tilde{\varphi}$ and masks $m_0(\chi), \tilde{m}_0(\chi)$ respectively. Assume that there exist periodic functions m_1 and \tilde{m}_1 , ($\mathbf{l} \in GF(p^s)$, $\mathbf{l} \neq \mathbf{0}$), such that for any $\mathbf{a}_{-N} \cdots \mathbf{a}_{-1} \in GF(p^s)$ and for any $\chi \in (F_{-N}^{(s)})^{\perp}$

$$\sum_{\mathbf{a}_0 \in GF(p^s)} m_{\mathbf{k}}(\chi \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \cdots \mathbf{r}_0^{\mathbf{a}_0}) \overline{m_1(\chi \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \cdots \mathbf{r}_0^{\mathbf{a}_0})} = \delta_{\mathbf{k}, \mathbf{l}}. \quad (6.1)$$

Define wavelets $\psi^{(\mathbf{l})}$ and $\tilde{\psi}^{(\mathbf{l})}$ by the equations

$$\hat{\psi}^{(\mathbf{l})}(\chi) = m_{\mathbf{l}}(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \quad \hat{\tilde{\psi}}^{(\mathbf{l})}(\chi) = \tilde{m}_{\mathbf{l}}(\chi)\hat{\tilde{\varphi}}(\chi\mathcal{A}^{-1}), \quad (6.2)$$

Theorem 6.1 ([5]) *Let φ and $\tilde{\varphi}$ be the scaling functions for dual MRAs and $\psi_{\mathbf{l}}, \tilde{\psi}_{\mathbf{l}}, \mathbf{l} \in GF(p^s)$ be the associated wavelets satisfying the matrix condition (6.1). Then the collections*

$$\{\psi_{n,h}^{(\mathbf{l})} = p^{\frac{ns}{2}}\psi^{(\mathbf{l})}(\mathcal{A}^n x - h) : h \in H_0, n \in \mathbb{Z}\}$$

and

$$\{\tilde{\psi}_{n,h}^{(\mathbf{l})} = p^{\frac{ns}{2}}\tilde{\psi}^{(\mathbf{l})}(\mathcal{A}^n x - h) : h \in H_0, n \in \mathbb{Z}\}$$

are biorthogonal. If addition

$$|\hat{\varphi}((K_n^+)^\perp \setminus (K_{n-1}^+)^\perp)| \leq \frac{C}{(1 + p^{ns})^{\frac{1}{2} + \varepsilon}}, \quad |\hat{\tilde{\varphi}}((K_n^+)^\perp \setminus (K_{n-1}^+)^\perp)| \leq \frac{C}{(1 + p^{ns})^{\frac{1}{2} + \varepsilon}},$$

$$|\hat{\psi}^{(\mathbf{l})}((K_n^+)^\perp)| \leq Cp^{ns}, \quad |\hat{\tilde{\psi}}^{(\mathbf{l})}((K_n^+)^\perp)| \leq Cp^{ns},$$

for some constant $C > 0, \varepsilon > 0$, then systems $\{\psi_{n,h}^{(\mathbf{l})}\}$ and $\{\tilde{\psi}_{n,h}^{(\mathbf{l})}\}$ form Riesz bases for $L_2(K)$.

Now we can continue to construct wavelets. Let T be N -valid tree. Using RF-algorithm we construct functions $m_0(\chi)$, $\hat{\varphi}(\chi)$ and set

$$\tilde{m}_0(\chi) = \begin{cases} 0, & m_0(\chi) = 0 \\ \frac{1}{m_0(\chi)}, & m_0(\chi) \neq 0 \end{cases}, \quad \hat{\tilde{\varphi}}(\chi) = \prod_{n=0}^{\infty} \tilde{m}_0(\chi\mathcal{A}^{-n}).$$

It is evident $\hat{\varphi}(\chi)\overline{\hat{\tilde{\varphi}}(\chi)} = 1$. Define functions

$$m_{\mathbf{l}}(\chi) = m_0(\chi\mathbf{r}_0^{-1}), \quad \tilde{m}_{\mathbf{l}}(\chi) = \tilde{m}_0(\chi\mathbf{r}_0^{-1}).$$

Lemma 6.1 *The following properties are true*

- 1) $m_{\mathbf{l}}(\tilde{E}_X\mathbf{r}_0^{\mathbf{l}}) \neq 0, \tilde{m}_{\mathbf{l}}(\tilde{E}_X\mathbf{r}_0^{\mathbf{l}}) \neq 0$ for any $\mathbf{l} \in GF(p^s)$.
- 2) $m_{\mathbf{l}}(\tilde{E}_X\mathbf{r}_0^{\mathbf{a}}) = \tilde{m}_{\mathbf{l}}(\tilde{E}_X\mathbf{r}_0^{\mathbf{a}}) = 0$ for $\mathbf{l} \neq \mathbf{a}$.
- 3) $m_{\mathbf{l}}(E) = \tilde{m}_{\mathbf{l}}(E) = 0$ for $\mathbf{l} \neq \mathbf{0}$.
- 4) $m_{\mathbf{l}}(\chi)m_{\mathbf{k}}(\chi) = \tilde{m}_{\mathbf{l}}(\chi)\tilde{m}_{\mathbf{k}}(\chi) = 0$ for $\mathbf{k} \neq \mathbf{l}$.

Proof. 1) If $\mathbf{l} \neq \mathbf{0}$ then $m_{\mathbf{l}}(\tilde{E}_X \mathbf{r}_0^{\mathbf{l}}) = m_0(\tilde{E}_X \mathbf{r}_0^{-\mathbf{l}} \mathbf{r}_0^{\mathbf{l}}) = m_0(\tilde{E}_X) \neq 0$.

2) Let $\chi \in \tilde{E}$ and $\chi = \chi_{-N} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}} \mathbf{r}_0^{\mathbf{a}_0}$, $\chi_{-N} \in (K_{-N}^+)^{\perp}$. It means that $m_0(\chi_{-N} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}} \mathbf{r}_0^{\mathbf{a}_0}) \neq 0$. Therefore if $\mathbf{a} \neq \mathbf{l}$, then

$$m_{\mathbf{l}}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}} \mathbf{r}_0^{\mathbf{a}_0} \mathbf{r}_0^{\mathbf{a}}) = m_0((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-1}^{\mathbf{a}_{-1}} \mathbf{r}_0^{\mathbf{a}_0 - \mathbf{l} + \mathbf{a}}) = 0,$$

since there cannot be two different paths from the node \mathbf{a}_{-N} to the root.

3) It follows from property 2) that $m_{\mathbf{l}}(\tilde{E}_X) = m_{\mathbf{l}}(\tilde{E}_X \mathbf{r}_0^{\mathbf{0}}) = 0$ for $\mathbf{l} \neq \mathbf{0}$.

4) If $\chi \in \tilde{E}$ then

$$m_{\mathbf{l}}(\chi) = m_0(\chi_{-N} \mathbf{r}_{-N}^{\alpha_{-N}} \dots \mathbf{r}_{-1}^{\alpha_{-1}} \mathbf{r}_0^{\alpha_0 - \mathbf{l}}),$$

$$m_{\mathbf{k}}(\chi) = m_0(\chi_{-N} \mathbf{r}_{-N}^{\alpha_{-N}} \dots \mathbf{r}_{-1}^{\alpha_{-1}} \mathbf{r}_0^{\alpha_0 - \mathbf{k}}),$$

where $\chi_{-N} \in (K_{-N}^+)^{\perp}$. Since there cannot be two different paths from the node \mathbf{a}_{-N} to the root we see that $m_{\mathbf{l}}(\chi) = 0$, or $m_{\mathbf{k}}(\chi) = 0$. \square

Define functions $\psi^{(\mathbf{l})}$ and $\tilde{\psi}^{(\mathbf{l})}$ by equations (6.2).

Theorem 6.2 1) The collectins $(\psi_{n,h}^{(\mathbf{l})})$ and $(\tilde{\psi}_{n,h}^{(\mathbf{l})})$ are biorthogonal.

2) The systems $(\psi_{n,h}^{(\mathbf{l})})$ and $(\tilde{\psi}_{n,h}^{(\mathbf{l})})$ form Riesz bases for $L_2(K)$.

Proof. Check equality (6.1). By Lemma 6.1, so that $m_{\mathbf{l}} m_{\mathbf{k}} = 0$ when $\mathbf{k} \neq \mathbf{l}$. Therefore, it suffices to prove the equation

$$\sum_{\alpha_0 \in GF(p^s)} m_{\mathbf{l}}(\chi_{-N} \mathbf{r}_{-N}^{\alpha_{-N}} \dots \mathbf{r}_{-1}^{\alpha_{-1}} \mathbf{r}_0^{\alpha_0}) \overline{\tilde{m}_{\mathbf{l}}(\chi_{-N} \mathbf{r}_{-N}^{\alpha_{-N}} \dots \mathbf{r}_{-1}^{\alpha_{-1}} \mathbf{r}_0^{\alpha_0})} = 1. \quad (6.3)$$

Since there cannot be two different paths from the node \mathbf{a}_{-N} to the root, it follows that (6.3) includes only one non-zero term is equal to one. By lemma 6.1 $m_{\mathbf{l}}(E) = 0$ for $\mathbf{l} \neq \mathbf{0}$. Therefore $\psi^{(\mathbf{l})}((K_{-N}^+)^{\perp}) = 0$.

Since $\text{supp } \hat{\varphi}(\chi) \subset (K_{H-2N+1}^+)^{\perp}$, it follows that $\text{supp } \hat{\psi}^{(\mathbf{l})}(\chi) \subset (K_{H-2N+2}^+)^{\perp}$. By analogy, $\tilde{\psi}^{(\mathbf{l})}((K_{-N}^+)^{\perp}) = 0$ and $\text{supp } \hat{\tilde{\psi}}^{(\mathbf{l})}(\chi) \subset (K_{H-2N+2}^+)^{\perp}$. So all conditions of theorem 6.1 is fulfilled, and theorem 6.2 is proved. \square

Finally we can write an algorithm to construct Riesz-wavelets.

W-algorithm.

- 1) Construct N -valid tree T using the basic steps.
- 2) Construct the mask $m_0(\chi)$ and refinable function $\varphi(\chi)$ using RF-algorithm.
- 3) Define functions $m_{\mathbf{l}}(\chi) = m_0(\chi \mathbf{r}_0^{-\mathbf{l}})$.
- 4) Set $\hat{\psi}^{(\mathbf{l})}(\chi) = m_{\mathbf{l}}(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1})$.
- 5) Find wavelets $\psi^{(\mathbf{l})}(\chi)$ using inverse Fourier transform.

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References

- [1] Huikun Jiang, Dengfeng Li, and Ning Jin. Multiresolution analysis on local fields. *J. Math. Anal. Appl.* 294 (2004) 523–532.
- [2] Dengfeng Li, Huikun Jiang. The necessary condition and sufficient conditions for wavelet frame on local fields. *J. Math. Anal. Appl.* 345 (2008) 500–510.
- [3] Biswaranjan Behera, Qaiser Jahan. Wavelet packets and wavelet frame packets on local fields of positive characteristic. *J. Math. Anal. Appl.* N 395, (2012), 1–14.
- [4] Biswaranjan Behera, Qaiser Jahan. Multiresolution analysis on local fields and characterization of scaling functions. *Adv. Pure Appl. Math.* N 3, (2012), 181–202.
- [5] Biswaranjan Behera, Qaiser Jahan. Biorthogonal Wavelets on Local Fields of Positive Characteristic. *Comm. in Math. Anal.* V.15, N.2, 52–75 (2013).
- [6] Taibleson M. H. *Fourier Analysis on Local Fields*, Princeton University Press, 1975.
- [7] V Yu Protasov, Y. A. Farkov. Dyadic wavelets and refinable functions on a half-line *Sbornik: Mathematics*(2006), 197(10):1529.
- [8] Y. A. Farkov, Orthogonal wavelets with compact support on locally compact abelian groups, *Izvestiya RAN: Ser. Mat.*, vol. 69, no. 3, pp. 193-220, 2005, English transl., *Izvestiya: Mathematics*, 69: 3 (2005), pp. 623-650.
- [9] Y. A. Farkov, Orthogonal wavelets on direct products of cyclic groups, *Mat. Zametki*, vol. 82, no. 6, pp. 934-952, 2007, English transl., *Math. Notes*: 82: 6 (2007).
- [10] Yu. A. Farkov, “Biorthogonal Wavelets on Vilenkin Groups”, *Selected topics of mathematical physics and p-adic analysis*, Collected papers, *Tr. Mat. Inst. Steklova*, 265, MAIK Nauka/Interperiodica, Moscow, 2009, 110–124
- [11] S.F.Lukomskii, Step refinable functions and orthogonal MRA on p -adic Vilenkin groups. *JFAA*, 2014, 20:1,pp.42-65.

- [12] S.F.Lukomskii. Multiresolution Riesz analysis on Vilenkin groups. Doklady Akademii Nauk, V. 457, 1, (2014), pp. 24–27.
- [13] S. F. Lukomskii. Riesz multiresolution analysis on zero-dimensional groups. Izvestiya: Mathematics, 2015, 79:1, 145–176
- [14] S.F.Lukomskii, A.M.Vodolazov Non-Haar MRA on local fields of positive characteristic J. Math. Anal. Appl. 433 (2016) 1415–1440
- [15] S.V. Kozyrev. Wavelet analysis as a p-adic spectral analysis. Izvestiya: Mathematics, 2002, 66:2, 367–376.
- [16] A.Yu. Khrennikov, V.M. Shelkovich, M. Skopina. p -adic refinable functions and MRA-based wavelets. J. Approx. Theory. 161:1, 2009, 226-238.
- [17] S. Albeverio, S. Evdokimov, M. Skopina. p-Adic Multiresolution Analysis and Wavelet Frames, J Fourier Anal Appl, (2010), 16: 693-714.
- [18] Y. A. Farkov, E.A.Rodionov. Algorithms for Wavelet Construction on Vilenkin Groups. p-Adic Numbers, Ultrametric Analysis and Applications, 2011, Vol. 3, No. 3, pp. 181–195.
- [19] S. F. Lukomskii and G. S. Berdnikov. N-Valid trees in wavelet theory on Vilenkin groups. International Journal of Wavelets, Multiresolution and Information Processing, Vol. 13, No. 5 (2015),
- [20] S. F. Lukomskii, G. S. Berdnikov, Iu. S. Kruss, On the orthogonality of a system of shifts of the scaling function on Vilenkin groups. Short Communications, Mathematical Notes, July 2015, Volume 98, Issue 1, pp 339-342